LESSON 19 - STUDY GUIDE

Abstract. In this lesson we focus on pointwise convergence of Fourier series. In the first part, we study pointwise convergence of summability methods, which have better properties and therefore stronger theorems. In the second part we concentrate on the convergence of partial sums, proving some of the most celebrated classical theorems for convergence of Fourier series at a point.

1. Pointwise convergence of Fourier series.

Study material: For the first part, we will closely follow the section **3 - Pointwise Convergence of** $\sigma_n(f)$ from chapter **I - Fourier Series on** $\mathbb{T}$, corresponding to pgs. 17–21 in the second edition [1] and pgs. 18–22 in the third edition [2] of Katznelson's book. In the second half, we cover part of the section **2 - Convergence and Divergence at a Point** from chapter **II - The Convergence of Fourier Series**, corresponding to pgs. 52–54 in the second edition [1] and pgs. 73–75 in the third edition [2].

For this lesson we will finally look at pointwise convergence of Fourier series. Pointwise convergence results in the theory of Fourier series tend to be the most difficult to prove and the deepest. The reason should now be obvious, because the most robust form of pointwise summation of trigonometric series is absolute convergence, which does not depend on the oscillating exponentials and therefore is equivalent for all points $t \in \mathbb{T}$. However, in spite of its optimal and ideal appeal, we saw in the previous lesson that absolutely convergent Fourier series correspond only to a very small subset of the continuous functions on $\mathbb{T}$. Therefore, if we want to address the general question of pointwise convergence of Fourier series of arbitrary functions in $L^1(\mathbb{T})$ we certainly need to consider mostly conditionally convergent series, where the oscillations of complex exponentials play an important role, and the convergence might change from point to point.

For such an unstable form of convergence it is no wonder that proofs are hard. For example, if we look at the general theorem for the convergence of convolutions with approximate identities, Theorem 1.2 in Lesson 10, we could only prove pointwise convergence for continuous functions there, in which case it is uniform. And, as we already know, pointwise convergence almost everywhere cannot be extracted from the convergence in L^p norm, as this only guarantees it for some subsequence. We will see, as we proceed further along the course, that pointwise convergence proofs do exist for this general setting of convolutions with approximate identities, but they require much more sophisticated machinery, in particular the use of the Hardy-Littlewood maximal function. So, if not even for approximate identities are there relatively straightforward proofs of general pointwise convergence, then, should the same type of results hold for partial sums of Fourier series, one can only expect them to be much more difficult to establish because they correspond to convolution with the Dirichlet kernel.

1.1. Pointwise convergence of Cesàro and Abel means.

We will start, thus, by studying pointwise convergence of summability methods of Fourier series. As we saw in Lesson 15, summability methods have much better convergence properties than partial sums, because they are convolutions with approximate identities, and one should expect stronger pointwise convergence results as well, even though they do not directly follow from the general properties seen before, in Lesson 10.

Date: May 5, 2020.

One such result is due to Fejér, obtained by exploiting specific properties of the kernel that today carries his name.

Theorem 1.1. (Fejér) *Let $f \in L^1(\mathbb{T})$. Then*

(1) *If the limit $\lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2}$ exists, including the possibility of being $+\infty$ or $-\infty$, then, the Cesàro means converges at t_0 to this value*

$$\lim_{N\to\infty} \sigma_N(f)(t_0) = \lim_{N\to\infty} K_N * f(t_0) = \lim_{h\to 0} \frac{f(t_0+h)+f(t_0-h)}{2}.$$

In particular, if f is continuous at t_0, then $\sigma_N(f)(t_0) \to f(t_0)$ when $N \to \infty$.

(2) *If f is continuous on a closed subset $F \subset \mathbb{T}$ then $\sigma_N(f)(t)$ converges uniformly to $f(t)$ for $t \in F$.*

(3) *If $m \le f(t)$ a.e. $t \in \mathbb{T}$ then $m \le \sigma_N(f)(t)$ for all $t \in \mathbb{T}$ and $N \in \mathbb{N}$. Likewise, if $f(t) \le M$ a.e. $t \in \mathbb{T}$ then $\sigma_N(f)(t) \le M$.*

Proof. The Fejér kernel is an approximate identity made up of positive even functions. Besides, they satisfy a stronger supremum (L^∞) condition than condition (3) in Definition 1.1, in Lesson 11. In fact, the Fejér kernel satisfies, for any $0 < \delta < \pi$

$$\lim_{N\to\infty} (\sup_{|t|\ge\delta} K_N(t)) = 0, \tag{1.1}$$

which implies the weaker L^1 form $\lim_{N\to\infty} \int_{|t|\ge\delta} |K_N(t)|dt = 0$ in that definition.

(1) Following Katznelson, we denote by $\check{f}(t_0) = \lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2}$ and when it is finite, we have

$$\begin{aligned}
\sigma_N(f)(t_0) - \check{f}(t_0) &= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s) f(t_0-s)ds - \check{f}(t_0) \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s) f(t_0-s)ds - \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s)ds\check{f}(t_0) \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s)\left[f(t_0-s) - \check{f}(t_0)\right]ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(s)\left[f(t_0-s) - \check{f}(t_0)\right]ds.
\end{aligned}$$

Now, using the fact that $K_N(t) = K_N(-t)$, we write this last integral as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(s)\left[f(t_0-s) - \check{f}(t_0)\right]ds = \frac{1}{\pi}\int_0^{\pi} K_N(s)\left[\frac{f(t_0+s)+f(t_0-s)}{2} - \check{f}(t_0)\right]ds,$$

and because $\check{f}(t_0) = \lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2}$, given arbitrarily small $\varepsilon > 0$, we can chose δ such that

$$|s| < \delta \Rightarrow \left|\frac{f(t_0+s)+f(t_0-s)}{2} - \check{f}(t_0)\right| < \varepsilon,$$

to break up the integral in two

$$\frac{1}{\pi}\int_0^{\delta} K_N(s)\left[\frac{f(t_0+s)+f(t_0-s)}{2} - \check{f}(t_0)\right]ds + \frac{1}{\pi}\int_{\delta}^{\pi} K_N(s)\left[\frac{f(t_0+s)+f(t_0-s)}{2} - \check{f}(t_0)\right]ds.$$

with the first bounded by ε while the second integral is made small by the property (1.1) of the Fejér kernel, for N sufficiently large, so that we finally have

$$|\sigma_N(f)(t_0) - \check{f}(t_0)| \le \varepsilon + \varepsilon\|f(\cdot) - \check{f}(t_0)\|_{L^1(\mathbb{T})}.$$

For the final case, when $\check{f}(t_0) = \lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2}$ is infinite, then one just has to look only at the integral

$$\sigma_N(f)(t_0) = \frac{1}{\pi}\int_0^{\pi} K_N(s)\left[\frac{f(t_0+s)+f(t_0-s)}{2}\right] ds,$$

and by splitting it in two integrals, the first from 0 to δ and the second from δ to π, in the same way as above, show that the first will become arbitrarily large as the infinite $\check{f}(t_0)$, when $N\to\infty$, while the second is bounded by $\varepsilon\|f\|_{L^1(\mathbb{T})}$ as before.

(2) This is totally analogous to part (3) of Theorem 1.2, in Lesson 11, for approximate identities, by noticing that F is compact for being a closed subset of a compact space $\mathbb{T}$. Therefore, from the Heine-Cantor theorem, continuity of f on F implies uniform continuity, and the proof proceeds from there exactly as in Theorem 1.2, by converting uniform continuity of f into uniform convergence of $K_N * f$ on F.

(3) This last property only uses the positivity and unit integral of the kernel, and not even the fact that it is an approximate identity. So, writing the difference $\sigma_N(f)(t) - m$ from the convolution with the kernel

$$\begin{aligned}
\sigma_N(f)(t) - m &= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s)f(t-s)ds - m \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s)f(t-s)ds - \frac{1}{2\pi}\int_{\mathbb{T}} mK_N(s)ds \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} K_N(s)\left[f(t-s)-m\right]ds \geq 0,
\end{aligned}$$

and likewise for the difference $M - \sigma_N(f)(t)$.

□

A few observations are in order. The first one is that the proof of Fejér's theorem depends only on the fact that the approximate identity kernel is positive, even and satisfies the stronger condition (1.1). So the same conclusion holds for any other approximate identity kernel that satisfies these same extra conditions. That is the case with the Poisson kernel used for Abel summability. Therefore we also have the following.

Corollary 1.2. *The same result as in the previous theorem holds if, instead of Cesàro means and the Fejér kernel, one uses Abel means and the Poisson kernel, respectively.*

Another important conclusion has to do with functions of bounded variation. It is well known, from basic measure theory, that BV functions on intervals of the real line can always be written as differences of two increasing functions that can have, at the most, a countable number of jump discontinuities, and for which lateral limits always exist at every point. So if $f \in BV(\mathbb{T})$ then, the limit $\lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2} = \dfrac{f(t_0+)+f(t_0-)}{2}$ always exists at every $t_0 \in \mathbb{T}$, where $f(t_0+)$ and $f(t_0-)$ are the right and left limits at t_0, respectively.

Corollary 1.3. *Let $f \in BV(\mathbb{T})$. Then, for every $t_0 \in \mathbb{T}$*

$$\lim_{N\to\infty} \sigma_N(f)(t_0) = \frac{f(t_0+)+f(t_0-)}{2}.$$

Of course, if the lateral limits $f(t_0+)$ and $f(t_0-)$ exist, then the limit $\lim_{h\to 0} \dfrac{f(t_0+h)+f(t_0-h)}{2}$ will also exist and equals the average of the two. But the converse is not necessary and might not be true. So we can have, say, an odd function like $f(t) = \sin(1/t)$ which is $L^1(]-\pi,\pi]) = L^1(\mathbb{T})$ and for which the lateral limits $f(0+)$ and $f(0-)$ do not exist. Nevertheless, because $f(t) = -f(t)$ then $\lim_{h\to 0} \frac{f(0+h)+f(0-h)}{2} = 0$ and so Fejér's theorem, still aplies here at $t_0 = 0$, with $\lim_{N\to 0} \sigma_N(f)(0) = 0$. Of course, this particular case is trivial because $K_N(t)$ is even, while $f(t)$ is odd so that $\sigma_N(f)(0)$ actually is zero for all N, but slightly more complicated examples could still as easily be constructed.

A final important conclusion that follows from Fejér's theorem is just a consequence of the elementary fact that, if a sequence converges, then the corresponding sequence of its arithmetic means also converges to the same limit. It is the pointwise result concerning the necessary value of the limit of partial sums of Fourier series, analogous to Corollary 1.2 in Lesson 15: if we know beforehand that the partial sums of the Fourier series converge, then the limit necessarily has to be the same as that of the Cesàro means.

Corollary 1.4. *Let $f \in L^1(\mathbb{T})$ such that the limit $\lim_{h\to 0} \frac{f(t_0+h)+f(t_0-h)}{2}$ exists. Then, if the Fourier series converges at t_0 its sum is necessarily equal to this limit. In particular, if f is continuous at $t_0 \in \mathbb{T}$, and the Fourier series of f converges at that point, then its sum necessarily equals $f(t_0)$.*

Now, in spite of its simplicity, Fejér's condition that the limit

$$\lim_{h\to 0} \frac{f(t_0+h)+f(t_0-h)}{2} = \check{f}(t_0) \tag{1.2}$$

exists is not very robust, for we can easily change f on a set of points in $\mathbb{T}$ with zero Lebesgue measure, which does not affect the Fourier coefficients, and consequently does not change the Cesàro means $\sigma_N(f)$ anywhere, but could easily disrupt the existence of this limit at any point $t_0 \in \mathbb{T}$, making Fejér's theorem completely useless. So, a condition in terms of integration of f, and not on pointwise values, that remains invariant under modifications of the function on sets of zero measure, should be better suited. And in fact, Fejér's condition (1.2) implies that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \left| \frac{f(t_0+s)+f(t_0-s)}{2} - \check{f}(t_0) \right| ds = 0. \tag{1.3}$$

This condition, called Lebesgue condition, is therefore more general than Fejér's and is robust under modifications of the function a.e. in $\mathbb{T}$. In fact, we will see later in the course, that this condition holds with $\check{f}(t_0) = f(t_0)$ at almost every $t_0 \in \mathbb{T}$. The points where that happens are called the Lebesgue points of f. By developing the machinery of maximal operators precisely with the goal of proving pointwise almost everywhere convergence of sequences of L^p functions and approximate identities, we will be able to establish quite general theorems. For now, we will simply state Lebesgue's pointwise convergence theorem of the Cesàro means under condition (1.3).

Theorem 1.5. (Lebesgue) *Let $f \in L^1(\mathbb{T})$ satisfy (1.3) at a point $t_0 \in \mathbb{T}$, for some value $\check{f}(t_0)$. Then $\lim_{N\to\infty} \sigma_N(f)(t_0) = \check{f}(t_0)$. In particular, knowing that the Lebesgue condition is actually satisfied a.e. $t_0 \in \mathbb{T}$ with $\check{f}(t_0) = f(t_0)$, we have that $\lim_{N\to\infty} \sigma_N(f)(t) = f(t)$ a.e. $t \in \mathbb{T}$.*

Proof. We will prove a general version of these types of results later in the course, once we develop the theory of maximal operators. A direct proof of this theorem can be found in Katznelson [1] pg.20, or [2] pg.21. □

Interestingly, this theorem now yields a very powerful uniqueness result based on pointwise convergence of Fourier series, again from the fact that if the limit of a sequence exists, than it has to be the same as that of its arithmetic means, combined with the almost everywhere aspect of Lebesgue's condition. So just as Corollary 1.4 above follows immediately from Fejér's theorem, we now obtain the following much stronger result as a consequence of Lebesgue's theorem. It is the pointwise almost everywhere analogue of Corollary 1.2 in Lesson 15.

Corollary 1.6. *Let $f \in L^1(\mathbb{T})$ be such that its Fourier series converges almost everywhere on a set of positive measure $E \subset \mathbb{T}$. Then its sum coincides with f almost everywhere on that set. In particular, if a Fourier series converges to zero almost everywhere on $\mathbb{T}$, then necessarily it is the Fourier series of the zero function. Or still, if two Fourier series converge pointwise almost everywhere on $\mathbb{T}$ to the same values, then they are the Fourier series of the same function (as equivalence classes in L^1) and necessarily their coefficients are all equal.*

We have thus established uniqueness of representation by pointwise almost everywhere convergent Fourier series. In Corollary 1.2 in Lesson 15 we did not write the analogue of this second, uniqueness representation, part of the statement, even though it was similarly true there. The reason was that we proved Proposition 1.3 right afterwards, in that same lesson, by using a different method not related to the convergence of Cesàro means, in which we obtained a much broader uniqueness result for the representation of functions by *any trigonometric series* converging in $L^1(\mathbb{T})$, with the coefficients necessarily having to be the Fourier coefficients of the function to which the series converges. Here, though, for pointwise almost everywhere convergence, we can only establish uniqueness of representation among *Fourier series*, because in fact uniqueness does not hold for general trigonometric series: there exist nonzero trigonometric series that converge almost everywhere on $\mathbb{T}$ to zero.

It is worth pausing for a second at this point to discuss the question of uniqueness of representation of functions by trigonometric series with a bit more depth than we have done, for a couple of times, in previous lessons. This problem was first raised by Riemann, in his Habilitationsschrift on Fourier and trigonometric series where, among other things, he developed the definition of what we now call the Riemann integral, to rigoroulsy compute the Fourier coefficients of very general functions, and proved a first version of the Riemann-Lesbesgue Lemma for his definition of integral. With a breakthrough idea, instead of asking which properties should a function possess to be representable by its Fourier series, he inverted the problem to the question of asking which properties should a trigonometric series posses in order to represent a function. The set of objects of study becomes considerably larger, for, recall from Lesson 17 for example, that there exist trigonometric series converging at every point $t \in \mathbb{T}$ to functions which are not in $L^1(\mathbb{T})$ and for which it does not even make sense to talk about Fourier series. The whole field became known as Riemann's theory of trigonometric series and the particular problem of uniqueness of representation of functions by trigonometric series became known as Riemann's uniqueness problem (Zygmund's book [3], for example, has a whole chapter, **IX - Riemann's Theory of Trigonometric Series** dedicated to this area).

A set $E \subset \mathbb{T}$ is called a *set of uniqueness*, or a U-set, if pointwise convergence of a trigonometric series in its complement implies uniqueness, i.e. if $\sum_{-\infty}^{\infty} c_n e^{int} = 0$ for $t \notin E$ implies $c_n = 0$. If that is not the case, that pointwise convergence in the complement of E does not guarantee uniqueness of the coefficients, i.e. if there are nonzero trigonometric series converging to zero for $t \notin E$, then the set E is said to be a *set of multiplicity*, an M-set, or a Menshov set. Of course, the larger E is, the harder it should be to guarantee uniqueness by pointwise convergence outside of it.

Riemann himself proved that the empty set $E = \emptyset$ is a set of uniqueness. That is the same as saying that if a trigonometric series converges pointwise *for all* $t \in \mathbb{T}$ then its coefficients are unique. Riemann's proof uses some properties of integration of Fourier series term by term and can be found in Zygmund's book [3], in the same chapter mentioned above, page 326.

Theorem 1.7. *Let $\sum_{n=-\infty}^{\infty} c_n e^{int}$ be a trigonometric series such that, for all $t \in \mathbb{T}$ the sequence of its partial sums converges to zero. Then, $c_n = 0$ for all $n \in \mathbb{Z}$.*

Georg Cantor then famously pursued this same problem and proved that for E countable and closed, uniqueness still holds. It was in the process of studying these questions that Cantor developed the theory of sets and cardinality for which he became known. Finally, Menshov, in 1916, provided an example of a nonempty set of multiplicity with zero measure, effectively destroying the conjecture that pointwise almost everywhere convergence of trigonometric series would be enough to ensure the uniqueness of the coefficients. So there do exist examples of nonzero trigonometric series converging pointwise almost everywhere to zero, which of course are not Fourier series, and they can be added to any other Fourier series to yield nonunique trigonometric series representing the same function almost everywhere.

To finish this section, we mention that a pointwise almost everywhere convergence result is equally true for the Abel means, with the Poisson kernel. But actually, it can proved with an even weaker condition than Lebesgue's (1.3),

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\left(\frac{f(t_0+s)+f(t_0-s)}{2}-\check{f}(t_0)\right)ds=0, \tag{1.4}$$

which actually holds with $\check{f}(t_0)=f(t_0)$ whenever the function equals the derivative of the integral, and that we know is almost everywhere from measure theory.

Theorem 1.8. **(Fatou)** *Let $f\in L^1(\mathbb{T})$ satisfy (1.4) at a point $t_0\in\mathbb{T}$, for some value $\check{f}(t_0)$. Then*

$$\lim_{r\to 1}P_r*f(t_0)=\lim_{r\to 1}\sum_{n=-\infty}^{\infty}r^{|n|}\hat{f}(n)e^{int_0}=\check{f}(t_0).$$

In particular, this is true almost everywhere $t_0\in\mathbb{T}$, with $\check{f}(t_0)=f(t_0)$.

Proof. Again see Katznelson [1] pg.21, or [2] pg.22. □

1.2. **Pointwise convergence of partial sums of Fourier series.** As pointed out at the beginning, if pointwise convergence of approximate identities, or even specific summability methods exploiting particular properties of concrete kernels like Fejér's or Poisson's, is not straightforward, then the same type of results for the much more badly behaved Dirichlet kernel can only be much harder to obtain. In fact, pointwise almost everywhere convergence of the partial sums of Fourier series, even for continuous functions, constitute some of the hardest and deepest problems in Fourier analysis, of which Carleson's theorem stands out as the most celebrated example.

For now, we will settle with some more elementary results on specific convergence at a general point. With that goal, we start by rewriting the partial sums in a more convenient form:

$$\sum_{n=-N}^{N}\hat{f}(n)e^{int}=D_N*f(t)=\frac{1}{2\pi}\int_{\mathbb{T}}D_N(t-s)f(s)ds=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(t-s)f(s)ds,$$

and now expanding the Dirichlet kernel to try to take advantage of specific properties

$$D_N(t)=\sum_{n=-N}^{N}e^{int}=\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}=\frac{\sin(Nt)\cos(\frac{t}{2})+\cos(Nt)\sin(\frac{t}{2})}{\sin\frac{t}{2}}=\frac{\sin Nt}{\tan\frac{t}{2}}+\cos(Nt), \tag{1.5}$$

we obtain

$$\sum_{n=-N}^{N}\hat{f}(n)e^{int}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin N(t-s)}{\tan\frac{t-s}{2}}f(s)ds+\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos N(t-s)f(s)ds. \tag{1.6}$$

Now, the second term here is easily seen to be unimportant, a sort of error term, as it converges uniformly to 0 as $N \to \infty$, for we can write it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos N(t-s) f(s)ds = \cos(Nt)\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(Ns)f(s)ds + \sin(Nt)\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(Ns)f(s)ds, \tag{1.7}$$

and while $|\cos(Nt)|, |\sin(Nt)| \leq 1$ uniformly for $t \in \mathbb{T}$, the integral terms converge to zero as $N \to \infty$ because of the Riemann-Lebesgue lemma. As for the first term in (1.6) we write it as

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin N(t-s)}{\tan\frac{t-s}{2}} f(s)ds = &\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin N(t-s)}{t-s} f(s)ds \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin N(t-s)\left(\frac{1}{\tan\frac{t-s}{2}} - \frac{2}{t-s}\right) f(s)ds,\end{aligned} \tag{1.8}$$

and we can now use the fact that

$$\left\|\frac{1}{\tan\frac{t-\cdot}{2}} - \frac{2}{t-\cdot}\right\|_{L^\infty(\mathbb{T})} = \left\|\frac{1}{\tan\frac{(\cdot)}{2}} - \frac{2}{(\cdot)}\right\|_{L^\infty(\mathbb{T})},$$

is independent of t to conclude that the function

$$\left(\frac{1}{\tan\frac{t-s}{2}} - \frac{2}{t-s}\right) f(s),$$

is in $L^1(\mathbb{T})$ in the s variable, uniformly for $t \in \mathbb{T}$. So that we can again argue as in (1.7) to conclude that this term also converges to 0 uniformly in $t \in \mathbb{T}$ as $N \to \infty$. Finally we use the fact that $\sin Nt/t$ is an even function to write the first term in (1.8) as

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin N(t-s)}{t-s} f(s)ds = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(Ns)}{s} f(t-s)ds = \frac{2}{\pi}\int_{0}^{\pi} \frac{\sin(Ns)}{s}\left(\frac{f(t+s)+f(t-s)}{2}\right) ds$$

so that, putting everything together, the partial sums of the Fourier series can be written more efficiently as

$$\sum_{n=-N}^{N} \hat{f}(n)e^{int} = \frac{2}{\pi}\int_{0}^{\pi} \frac{\sin(Ns)}{s}\left(\frac{f(t+s)+f(t-s)}{2}\right) ds + o(1), \tag{1.9}$$

where $o(1)$ denotes a term that converges to zero uniformly in $t \in \mathbb{T}$ as $N \to \infty$: the whole information about the convergence of the partial sums is encoded in this integral.

A first result that we can immediately extract from this formula is the localized character of the convergence of Fourier series. In spite of the Fourier coefficients being the result of integrals taken over the whole of $\mathbb{T}$, if two functions coincide on an open neighborhood of a point t_0 then their Fourier series either diverge or converge identically at t_0.

Proposition 1.9. (Riemann's Principle of Localization) *Let $f \in L^1(\mathbb{T})$ vanish in an open interval $I \subset \mathbb{T}$. Then its Fourier series converges to zero at every $t \in I$ and uniformly on compact subsets of I.*

Proof. From (1.9) we see that, if f vanishes on the open interval I and $t \in I$, then we have

$$\int_0^{\pi} \left|\frac{1}{s}\left(\frac{f(t+s)+f(t-s)}{2}\right)\right| ds < \infty,$$

and so, from the Riemann-Lebesgue lemma we conclude that the right-hand side of (1.9) converges to zero.

Besides, if $K \subset I$ is compact, and $t \in K$ then, the family of functions

$$\phi_t(s) = \frac{1}{s}\left(\frac{f(t+s)+f(t-s)}{2}\right),$$

form a compact subset of $L^1(\mathbb{T})$. And the Riemann-Lebesgue lemma then holds uniformly[1] for that compact subset of $L^1(\mathbb{T})$, implying uniform convergence to zero of the partial sums, for $t \in K$. □

Using the same type of ideas, the following is one of the most celebrated criteria for convergence of Fourier series at a point.

Theorem 1.10. (Dini's Test) *Let $f \in L^1(\mathbb{T})$ and suppose there exists a constant A such that*

$$\int_0^\pi \left| \frac{f(t+s)+f(t-s)}{2} - A \right| \frac{ds}{s} < \infty.$$

Then,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int} = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{int} = A.$$

Proof. The result follows simply by applying the same reasoning as in the localization principle to the function $g(t) = f(t) - A$. This criterion then implies that, at this fixed $t \in \mathbb{T}$ we have

$$\int_0^\pi \left| \frac{1}{s}\left(\frac{g(t+s)+g(t-s)}{2}\right) \right| ds < \infty,$$

and therefore that the Fourier series of g converges to zero at t, which is the same as the convergence of the Fourier series of f to A. □

A very important corollary of Dini's test is the uniform convergence of Fourier series of Hölder continuous functions. Recall, from Bernstein's theorem presented in the previous lesson, that only for Hölder-α, with $\alpha > 1/2$, could we guarantee absolute convergence. Uniform convergence, though, holds for any positive α.

Corollary 1.11. *Let $f \in C^{0,\alpha}(\mathbb{T})$ be Hölder-α continuous, for any $0 < \alpha \leq 1$. Then the Fourier series of f converges uniformly to it on $\mathbb{T}$.*

Proof. Hölder-α continuous functions satisfy

$$|f(t) - f(s)| \leq C|t-s|^\alpha,$$

for some $C \geq 0$ and all $t, s \in \mathbb{T}$. But this implies

$$\left| \frac{f(t+s)+f(t-s)}{2} - f(t) \right| \frac{1}{s} \leq \frac{|f(t+s)-f(t)|}{2s} + \frac{|f(t-s)-f(t)|}{2s} \leq \frac{C}{s^{1-\alpha}},$$

which is integrable in $s \in [0,\pi]$ independently of t. And we can therefore use Dini's test uniformly for all $t \in \mathbb{T}$ with $A = f(t)$ at every point. □

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[1] If $\mathcal{K} \subset L^1(\mathbb{T})$ is a compact family of functions, then, given arbitrarily small ε, $\mathcal{K}$ can be covered by a finite number of balls of radius ε centered at trigonometric polynomials $\{P_j\}_{1\leq j\leq n}$, because of their density in $L^1(\mathbb{T})$. Thus, every $f \in \mathcal{K}$ satisfies $\|f - P_j\|_{L^1(\mathbb{T})} < \varepsilon$ for some P_j. And picking the largest degree $N = \max_{1\leq j\leq n}(\text{degree } P_j)$ among these finite polynomials, then $|\hat{f}(n)| \leq \varepsilon$ for all $f \in \mathcal{K}$, and $|n| > N$, establishing the Riemann-Lebesgue property uniformly for all $\mathcal{K}$.

It should be emphasized how this set of results all hinge on properly understanding oscillatory integrals, of which the Riemann-Lebesgue lemma is the prototypical example that we have used repeatedly. At the root of all these theorems is the integral in (1.9)

$$\frac{2}{\pi}\int_0^\pi \frac{\sin(Ns)}{s}\left(\frac{f(t+s)+f(t-s)}{2}\right)ds,$$

which is obviously an oscillatory integral due to the presence of the $\sin(Ns)$ term. Pointwise convergence of Fourier series then boils down "simply" to showing that this integral converges to zero as the frequency of the oscillations N increases to infinity

Finally we notice that the presence of the averages $\frac{f(t+s)+f(t-s)}{2}$ is a consequence of the Dirichlet kernel being an even function with respect to the origin. This is something that had already been observed in the pointwise convergence of the summability methods, at the beginning of this lesson. Of course if the limit

$$\lim_{s\to 0}\frac{f(t+s)+f(t-s)}{2} = \check{f}(t),$$

exists, then necessarily the constant A in Dini's test will have to be $\check{f}(t)$, although the existence of this limit just by itself is not a sufficient condition for Dini's test to be satisfied. This is something that we already knew from Fejér's theorem at the beginning of the current lesson: if the limit exists, then the Cesàro means converge to it and, necessarily, so will the partial sums of the Fourier series, in case it converges. The problem rests, obviously, in the fact that, although the arithmetic means of a sequence always converges to the same limit if the original sequence converges, nevertheless the means might still converge with the sequence diverging.

As we already have at our disposal quite strong theorems establishing pointwise convergence of the Cesàro means, it would be extremely useful if we could somehow go in the reverse direction, and from the convergence of arithmetic means conclude the convergence of the original sequence. As this is generally false, extra conditions should be imposed on the sequence for such type of inverse results to hold: stronger conditions make the conclusion easier, while weaker assumptions can turn the proofs of these type of reverse results extremely hard. This kind of results, where convergence of sequences is obtained from convergence of their arithmetic means are called *Tauberian* theorems. We close today's lesson with an important Tauberian theorem due to Hardy, that relates the convergence of the Cesàro sums with the partial sums of the Fourier series.

Theorem 1.12. (Hardy's Tauberian theorem) *Let $f \in L^1(\mathbb{T})$ be such that its Fourier coefficients decays as n^{-1} i.e.*

$$|\hat{f}(n)| \le \frac{C}{n},$$

for some positive constant $C > 0$, when $|n| \to \infty$. Then the Cesàro means $\sigma_N(f)(t) = K_N * f(t)$ and the partial sums $S_N[f](t) = D_N * f(t)$ both converge on the same set of points $t \in \mathbb{T}$ and to the same values. If the convergence of the Cesàro means is uniform on some set, so will be the convergence of the partial sums.

Proof. The decay condition implies that, for every $\varepsilon > 0$ there exists a $\lambda > 1$ for which

$$\limsup_{N\to\infty} \sum_{N\le |n|\le \lambda N} |\hat{f}(n)| < \varepsilon.$$

The proof really only relies on this estimate. Of course, with the demanded decay, for this to work we just need to take $\lambda < 1+\varepsilon$, for large N.

The following relation then exists between the terms of the partial sums of the Fourier series and the terms of the Cesàro means,

$$S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int} = \frac{[\lambda N]+1}{[\lambda N]-N}\sigma_{[\lambda N]}(f)(t) - \frac{N+1}{[\lambda N]-N}\sigma_N(f)(t) - \frac{[\lambda N]+1}{[\lambda N]-N}\sum_{N<|n|\leq \lambda N}\left(1-\frac{n}{[\lambda N]+1}\right)\hat{f}(n)e^{int}, \tag{1.10}$$

where $[\lambda N]$ denotes the integer part of λN. From the previous condition, the last term on the right hand side, with the block of frequencies between N and λN of the Cesàro means, is smaller than ε for N sufficiently large. The first two terms can be grouped as

$$\frac{[\lambda N]\sigma_{[\lambda N]}(f)(t) - N\sigma_N(f)(t)}{[\lambda N]-N} + \frac{\sigma_{[\lambda N]}(f)(t) - \sigma_N(f)(t)}{[\lambda N]-N},$$

so that if we subtract $\sigma_N(f)(t)$ from both sides of (1.10) we get, for large N

$$\begin{aligned} |S_N[f](t) - \sigma_N(f)(t)| &\leq \left|\frac{[\lambda N]\sigma_{[\lambda N]}(f)(t) - N\sigma_N(f)(t)}{[\lambda N]-N} + \frac{\sigma_{[\lambda N]}(f)(t) - \sigma_N(f)(t)}{[\lambda N]-N} - \sigma_N(f)(t)\right| + \varepsilon \\ &\leq \left|\frac{[\lambda N]\big(\sigma_{[\lambda N]}(f)(t) - \sigma_N(f)(t)\big)}{[\lambda N]-N}\right| + \left|\frac{\sigma_{[\lambda N]}(f)(t) - \sigma_N(f)(t)}{[\lambda N]-N}\right| + \varepsilon, \end{aligned}$$

and thus, by making N even larger, this can be made smaller than 2ε.

Clearly, from the right-hand side of this estimate we can see that the difference $|S_N[f](t)-\sigma_N(f)(t)|$, in terms of its dependence on t, only changes with the pointwise rate of convergence of $\sigma_N(f)(t)$. Therefore, if the convergence of the Cesàro means is uniform for t on a particular subset of $\mathbb{T}$, so will be the convergence of the difference, from the above estimate. And thus of $S_N[f](t)$. □

One of the better known results about pointwise convergence of Fourier series follows from a direct application of Hardy's Tauberian theorem. For functions of bounded variation, we know from Theorem 1.4 in Lesson 17 that their Fourier coefficients decay precisely as n^{-1} while, on the other hand, from Corollary 1.3 above, we know that their Cesàro means converge at every point to the average of the lateral limits, which always exist. So an application of the previous theorem yields the following.

Corollary 1.13. *Let $f \in BV(\mathbb{T})$. Then, for every for every $t_0 \in \mathbb{T}$ the partial sums of the Fourier series of f converge and we have*

$$\lim_{N\to\infty} S_N[f](t_0) = \lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}(n)e^{int_0} = \frac{f(t_0+)+f(t_0-)}{2}.$$

In particular, if $f \in BV(\mathbb{T})$ is continuous at t_0 then its Fourier series converges to $f(t_0)$ and the convergence is uniform on compact subsets of continuity. And so, the Fourier series of absolutely continuous functions on the whole circle $\mathbb{T}$, i.e. $f \in C(\mathbb{T}) \cap BV(\mathbb{T})$, converge uniformly to f on $\mathbb{T}$

The well known theorem by Dirichlet, about the pointwise convergence of Fourier series of piecewise C^1 functions, usually taught in elementary Calculus courses on a first encounter with Fourier series, is a particular case of this one. And it can also be easily proved from Dini's test.

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